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On Restricted Centers of Sets

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Given a normed linear space X, a family $\mathscr{V}$ of nonempty closed subsets of X, and a family $\mathscr{F}$ of nonempty closed and bounded subsets of X, we identify three properties (R_1), (R_2), and (R_3) of the triplets $(X, V, \mathscr{F})$ where $V \in \mathscr{V}$, and two properties (R_4), $(\tilde{R}_4)$ of the triplets $(X, \mathscr{V}, \mathscr{F})$, with a view to studying existence of restricted centers and stability of the restricted center map. This leads to a sharpening of many known results as well as to some new results for existence of restricted centers, and it also enables us to obtain some new continuity results for restricted center maps.

INTRODUCTION

Let X be a normed linear space and V a nonempty subset of X. For a bounded subset F of X, let

$$r(F; x) := \sup\{\|x - y\| : y \in F\}$$

denote the radius of the smallest closed ball centered at x covering F and let

$$\operatorname{rad}_V(F) := \inf\{r(F; x) : x \in V\},$$

$$\operatorname{Cent}_V(F) := \{v_0 \in V : r(F; v_0) = \operatorname{rad}_V(F)\}.$$

The number $\operatorname{rad}_V(F)$ is called the *Chebyshev radius* of F in V and an element $v_0 \in \operatorname{Cent}_V(F)$ is called a *restricted center* (or a *best simultaneous approximation*) of F in V. When F is a singleton $\{x\}$, $x \in X$, then $\operatorname{rad}_V(F)$ is the distance of x from V, denoted by $\operatorname{dist}(x; F)$, and $\operatorname{Cent}_V(F)$ is the set

$$P_V(x) := \{v_0 \in V : \|x - v_0\| = \operatorname{dist}(x; V)\}$$

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of best approximations to x in V. It is obvious that

$$r(F, x) = r(\bar{F}; x) = r(\overline{\text{co}}\, F, x),$$

where $\overline{\text{co}}$ stands for the closed convex hull. Therefore the assumption on the bounded set F to be closed (and (or) convex) is not an additional restriction. The study of restricted centers, initiated by Garkavi [9], has attracted much attention. Questions concerning the existence, uniqueness, and stability of restricted centers have been analyzed by many authors (cf., e.g., [10, 11, 23, 24, 2, 15–17, 3–5, 21]). For a recent survey of results in this direction, we refer the reader to [6] (cf. also the earlier expository article [8]).

In this paper, for the most part, we deal with restricted centers of sets of continuous (and (or) bounded) vector-valued functions defined on a topological space T. We identify three properties (R_1), (R_2), and (R_3) of the triplets $(X, V, \mathcal{F})$, where V is a nonempty closed subset of a normed linear space X and $\mathcal{F}$ is a given family of nonempty closed and bounded subsets of X, with a view to study nonemptiness of $\text{Cent}_V(F)$ and continuity of the Cent_V-map: $F \to \text{Cent}_V(F)$ for $F \in \mathcal{F}$. Property (R_1) is a strengthened version of the $1\frac{1}{2}$-ball property of [25], property (R_2) strengthens both property (P_1) of [17] and property (A) of [20], and property (R_3) is a strengthened form of property (P_2) of [17]. We also identify two strengthened versions of property (R_2) which we call properties (R_4) and $(\tilde{R}_4)$, respectively, of the triplet $(X, \mathcal{V}, \mathcal{F})$, where X, $\mathcal{F}$ are as before and $\mathcal{V}$ is a given family of nonempty closed subsets of X. Property (R_4) yields equi-Hausdorff continuity and property $(\tilde{R}_4)$ yields equi-upper Hausdorff semicontinuity of the family $\{\text{Cent}_V : V \in \mathcal{V}\}$ of restricted center maps. We explore various examples of triplets satisfying properties (R_1)–(R_4) and analyze the interconnections between these and the other known related properties [25, 15, 17, 4, 20]. This leads us to obtain extensions of several known proximinality results (e.g., [7, Corollary 3.1; 20, Theorem 4; 13, Theorem 2.1]) to restricted centers. Some new continuity results, as well as some new results on continuous selection of restricted center maps are also obtained.

1. Preliminaries and Notation

In the sequel, X will be a normed linear space over the field $\mathcal{K} = \mathcal{R}$ or $\mathcal{C}$ and θ will denote the origin of X. The open (resp. closed) ball of center x_0 and radius $r > 0$ will be denoted by $B(x_0; r)$ (resp. $\bar{B}(x_0; r)$). $CL(X)$ (resp. $CB(X)$, resp. $K(X)$) will denote the class of nonempty closed (resp. nonempty closed and bounded, resp. nonempty compact) subsets of X.

$CC(X)$ will denote the class of nonempty closed and convex subsets of X. For sets A, B in $CL(X)$, let $h(A, B) := \sup\{\operatorname{dist}(a; B) : a \in A\}$ denote the *Hausdorff hemidistance* [12, p. 38] between A and B and let $H(A, B) := \max\{h(A, B), h(B, A)\}$ denote the *Hausdorff distance* between A and B. Hausdorff distance so defined yields an infinite valued metric on $CL(X)$. Restricted to any subfamily $\mathscr{F}$ of $CB(X)$, it defines a finite valued metric. We denote by τ_H the topology of Hausdorff metric on $\mathscr{F}$. If T is a topological space, then a set-valued map $\Gamma: T \to CL(X)$ is said to be *upper Hausdorff semicontinuous*, abbreviated u.H.s.c. (resp. *lower Hausdorff semicontinuous*, abbreviated l.H.s.c.) if for every $t_0 \in T$ and every $\varepsilon > 0$, there is a neighbourhood N of t_0 such that $h(\Gamma(t), \Gamma(t_0)) < \varepsilon$ (resp. $h(\Gamma(t_0), \Gamma(t)) < \varepsilon$) for each $t \in N$. Γ is said to be *Hausdorff continuous*, abbreviated H-continuous, if it is both u.H.s.c. and l.H.s.c. Recall that Γ is said to be *upper semicontinuous*, abbreviated u.s.c. (resp. *lower semicontinuous*, abbreviated l.s.c.) if $\Gamma^{-1}(t) := \{t \in T : \Gamma(t) \cap A \neq \phi\}$ is closed (resp. open) for each closed (resp. open) subset A of X. If Γ is both u.s.c. and l.s.c., then it is said to be *continuous*. It is well known (cf., e.g., [12, Theorem 7.1.11]), that if Γ is u.s.c., then it is u.H.s.c. and that if Γ is l.H.s.c., then it is l.s.c. Moreover, if Γ maps T into $K(X)$, then Γ is continuous if and only if it is H-continuous.

Given a set F in $CB(X)$ and $r > 0$, let

$$S_r(F) := \{x \in X : r(F; x) \leqslant r\} = \bigcap \{\bar{B}(y; r) : y \in F\}$$

denote the sublevel set of the function $r(F, \cdot)$ at height r. If $V \in CL(X)$ and $\mathscr{F} \subset CB(X)$, then V is said to satisfy the *restricted center property* (abbreviated r.c.p.) for $\mathscr{F}$ if $\operatorname{Cent}_V(F) \neq \phi$, for each $F \in \mathscr{F}$. We say that X *admits centers* for $\mathscr{F}$ if $\operatorname{Cent}_X(F) \neq \phi$ for each $F \in \mathscr{F}$. If V satisfies r.c.p. for $\mathscr{F}$, then a map $c: \mathscr{F} \to V$ such that $c(F) \in \operatorname{Cent}_V(F)$ (resp. $c(F) \in \operatorname{Cent}(F) := \operatorname{Cent}_X(F)$), for each $F \in \mathscr{F}$, is called a restricted *center selection* for V (resp. a *center selection* for X) defined on $\mathscr{F}$. If T is an arbitrary set (resp. a topological space) and U is a Banach space, then we denote by $l_\infty(T, U)$ (resp. $\mathscr{C}(T, U)$) the space of bounded (resp. continuous) U-valued functions on T. We equip $l_\infty(T, U)$ with the sup norm and denote by $\mathscr{C}_b(T, U)$ the space $\mathscr{C}(T, U) \cap l_\infty(T, U)$ equipped with the restricted norm in case T is a topological space. We record the following elementary fact useful in the sequel as a lemma.

LEMMA 1.1. *Let $X = \mathscr{C}_b(T, U)$, $F \in K(X)$, and for each $t \in T$ let $F(t)$ denote the set $\{f(t) : f \in F\}$; then the set-valued function $t \to F(t)$ of T into $K(U)$ is H-continuous, and hence also continuous.*

Proof. This follows immediately from the equicontinuity of F. ∎

A subset V of $\mathscr{C}_b(T, U)$ is said to be a $\mathscr{C}_b(T, \mathscr{K})$-submodule of $\mathscr{C}_b(T, U)$ if $\alpha f + \beta g \in V$ whenever f, g are in V and α, β are in $\mathscr{C}_b(T, \mathscr{K})$. A subset V of $l_\infty(T, U)$ is said to be a *convex* $l_\infty(T, [0, 1])$-*submodule* if $\alpha f + (1-\alpha) g \in V$ whenever f, g are in V and $\alpha \in l_\infty(T, [0, 1])$. In case T is compact Hausdorff, V is said to be a *Weierstrass-Stone subspace* of $\mathscr{C}_b(T, U)$ (abbreviated *W-S subspace*) if there is a compact Hausdorff space S and a continuous surjection $\pi: T \to S$ such that $V = \{g \circ \pi : g \in \mathscr{C}_b(S, U)\}$.

Recall [1] that a linear projection P in a Banach space X is called an *L-projection* if $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in X$. A linear subspace L of X is called an *L-summand* if it is the range of an L-projection in X and a closed subspace V of X is called an *M-ideal* if its annihilator $M^\perp$ is an L-summand in X^*. Last, recall that a (real) Banach space X is said to be a Lindenstrauss space if X^* is isometric to $L_1(\mu)$ for some measure μ. Lindenstrauss [14] has shown that this is equivalent to the property that every collection of pairwise intersecting closed balls in X whose centers form a compact set, has nonempty intersection.

2. Property (R_1)

We need to recall here the notion of $1\frac{1}{2}$-ball property introduced by Yost [25]. A closed subspace V of a Banach space X is said to satisfy the $1\frac{1}{2}$-*ball property* in X if $V \cap \bar{B}(x; r_1) \cap \bar{B}(y; r_2) \neq \phi$, whenever $x \in V$, $y \in X$, $r_1 > 0$, and $r_2 > 0$ are such that $V \cap \bar{B}(y; r_2) \neq \phi$ and $\|x - y\| < r_1 + r_2$. This property ensures proximinality of V and existence of a selection for the metric projection $P_V(\cdot)$, which is continuous, homogeneous, and quasi-additive [25]. For studying restricted centers, it is apparently more useful to introduce the following property which is a strengthened version of the $1\frac{1}{2}$-ball property. Although indirectly used in the proof of Proposition 3 of [17], it does not appear to have been well-studied elsewhere in the literature.

Definition 2.1. Given $V \in CL(X)$ and $\mathscr{F} \subset CB(X)$, the triplet $(X, V, \mathscr{F})$ is said to satisfy *property* (R_1) if $V \cap \bar{B}(x; r_1) \cap S_{r_2}(F) \neq \phi$, whenever $x \in V$, $F \in \mathscr{F}$, $r_1 > 0$, and $r_2 > 0$ are such that $V \cap S_{r_2}(F) \neq \phi$ and $r(F; x) < r_1 + r_2$. Clearly if V is a closed subspace of X, $\mathscr{F}$ contains all singletons in X and $(X, V, \mathscr{F})$ satisfies property (R_1), then V satisfies the $1\frac{1}{2}$-ball property in X.

Theorem 2.2. *Suppose X is a Banach space, $V \in CL(X)$, and $\mathscr{F} \subset CB(X)$. If $(X, V, \mathscr{F})$ satisfies property* (R_1), *then V satisfies r.c.p. for $\mathscr{F}$.*

Proof. Let F in $\mathscr{F}$ be given. We inductively construct a sequence v_n in V satisfying

$$\|v_n - v_{n+1}\| \leqslant 2^{-n} \tag{1}$$

and

$$r(F; v_n) \leqslant \operatorname{rad}_V(F) + 2^{-n} \tag{2}$$

Indeed, suppose $v_n \in V$ is given satisfying (2). Then

$$V \cap S_{\operatorname{rad}_V(F) + 2^{-n-1}}(F) \neq \phi$$

and $r(F; v_n) < \operatorname{rad}_V(F) + 2^{-n-1} + 2^{-n}$. By (R_1)

$$V \cap \bar{B}(v_n; 2^{-n}) \cap S_{\operatorname{rad}_V(F) + 2^{-n-1}}(F) \neq \phi.$$

Pick up v_{n+1} from the last set. Then v_{n+1} satisfies (1) and (2), and the induction is complete. By (1), $\{v_n\}$ is Cauchy and if $v = \lim_n v_n$, then $v \in V$ and by (2), $v \in \operatorname{Cent}_V(F)$. Thus $\operatorname{Cent}_V(F) \neq \phi$ and V satisfies r.c.p. for $\mathscr{F}$. ▌

In the following propositions we consider examples of triplets $(X, V, \mathscr{F})$ satisfying property (R_1).

PROPOSITION 2.3. *If V is an M-ideal in a Lindenstrauss space X, then the triplet $(X, V, K(X))$ satisfies property* (R_1).

Proof. Suppose $x \in X$, $F \in K(X)$, $r_1 > 0$, and $r_2 > 0$ be such that $V \cap S_{r_2}(F) \neq \phi$ and $r(F; x) < r_1 + r_2$. Then $\bar{B}(x; r_1) \cap \bar{B}(y; r_2) \neq \phi$ for each $y \in F$ and since $F \in K(X)$, by a theorem of Lindenstrauss [8, p. 62] we have $\bar{B}(x; r_1) \cap (\bigcap \{\bar{B}(y; r_2) : y \in F\}) \neq \phi$.

Since $V \cap \bar{B}(x; r_1) \neq \phi$ and $V \cap \bar{B}(y; r_2) \neq \phi$ for each $y \in F$ by [16, Lemma 2.1], $V \cap \bar{B}(x; r_1) \cap S_{r_2}(F) \neq \phi$. ▌

PROPOSITION 2.4. *If T is a compact Hausdorff space, U is a Lindenstrauss space, $X = \mathscr{C}(T, U)$, and V is a* W–S *subspace of X, then $(X, V, K(X))$ satisfies property* (R_1).

Proof. There are a compact Hausdorff space S and a continuous surjection $\pi: T \to S$ such that $V = \{g \circ \pi : g \in \mathscr{C}(S, U)\}$. Suppose $g \circ \pi \in V$, $F \in K(X)$, $r_1 > 0$, and $r_2 > 0$ are such that $V \cap (\bigcap \{\bar{B}(f; r_2) : f \in F\}) \neq \phi$ and $r(F; g \circ \pi) < r_1 + r_2$. Define $\Phi: S \to CC(U)$ with values

$$\Phi(s) := \bar{B}(g(s); r_1) \cap \left(\bigcap \{\bar{B}(f(t); r_2) : f \in F,\ t \in \pi^{-1}(s)\} \right).$$

By Lemma 1.1, the set-valued map $t \to F(t)$ of T into $K(U)$ is u.s.c. and therefore, by [12, Theorem 7.4.2] the set $\{f(t) : f \in F,\ t \in \pi^{-1}(s)\}$ is in $K(U)$. By the theorem of Lindenstrauss [8, p. 62], we have $\Phi(s) \neq \phi$ for each $s \in S$. We show that Φ is l.s.c. To this end suppose $\Phi(s_0) \cap O \neq \phi$ for a point $s_0 \in S$ and an open set O in U. Pick a point u in this set. Then $\|u - g(s_0)\| \leqslant r_1$, $f(\pi^{-1}(s_0)) \subset \bar{B}(u; r_2)$ for each $f \in F$ and $B(u, \varepsilon) \subset O$ for some $\varepsilon > 0$. It is easily seen that the set-valued map $s \to F(\pi^{-1}(s))$ of S into $K(U)$ is u.s.c. Hence there is a neighbourhood N_1 of s_0 such that $F(\pi^{-1}(s)) \subset B(u; r_2 + \varepsilon)$ for every $s \in N_1$ and by continuity of g, there is a neighbourhood N_2 of s_0 such that $\|g(s) - g(s_0)\| < \varepsilon$ for every $s \in N_2$. Taking $N = N_1 \cap N_2$, we have $\bar{B}(u; \varepsilon) \cap \bar{B}(f(t); r_2) \neq \phi$ and $\bar{B}(u; \varepsilon) \cap \bar{B}(g(s); r_1) \neq \phi$ for each $f \in F$, $t \in \pi^{-1}(s)$ and $s \in N$. Again by the same theorem of Lindenstrauss used before, we have $\Phi(s) \cap \bar{B}(u; \varepsilon) \neq \phi$ for each $s \in N$, which proves that Φ is l.s.c. By Michael's selection theorem [18, Theorem 3.2″], Φ has a continuous selection h. It is easily verified that $h \circ \pi \in V \cap \bar{B}(g \circ \pi; r_1) \cap S_{r_2}(F)$. ∎

PROPOSITION 2.5. *Let T, S, U, X, and π be as in the previous proposition. If E is a closed subset of S and $V = \{g \circ \pi : g \in \mathscr{C}(S, U)$ and $g|_E = 0\}$, then $(X, V, K(X))$ satisfies property* (R_1).

Proof. Let Φ be as in the proof of the previous proposition. Suppose $g \circ \pi \in V$, $F \in K(X)$, $r_1 > 0$, and $r_2 > 0$ are such that $V \cap S_{r_2}(F) \neq \phi$ and $r(F; g \circ \pi) < r_1 + r_2$. Pick up an element $\bar{g} \circ \pi \in V \cap S_{r_2}(F)$, then $\|f(t)\| \leqslant \|f(t) - \bar{g}(\pi(t))\| \leqslant r_2$ for each $t \in \pi^{-1}(E)$ and each $f \in F$. This shows that $\theta \in \Phi(s)$ for each $s \in E$. Define

$$\Phi_0(s) := \begin{cases} \phi(s), & s \notin E \\ \{\theta\}, & s \in E. \end{cases}$$

Then Φ_0 is l.s.c. and the existence of a continuous selection for Φ_0 shows that $V \cap \bar{B}(g \circ \pi; r_1) \cap S_{r_2}(F) \neq \phi$. ∎

COROLLARY 2.6. *If T is a compact Hausdorff space, $X = \mathscr{C}(T, \mathscr{R})$, and V is a closed subalgebra of X, then $(X, V, K(X))$ satisfies property* (R_1).

Proof. This follows from the Stone–Weierstrass theorem and Proposition 2.4 if V vanishes at no point of T and from Proposition 2.5 if V vanishes at some point of T. ∎

COROLLARY 2.7. *If T is a compact Hausdorff space, U is a Lindenstrauss space, $X = \mathscr{C}(T, U)$, E is a closed subset of T, and $V = \{g \in \mathscr{C}(T, U) : g|_E = \theta\}$, then $(X, V, K(X))$ satisfies property* (R_1).

PROPOSITION 2.8. *If T is a paracompact Hausdorff space, U is a Lindenstrauss space, V is an M-ideal in U, and $X = l_\infty(T, U)$, then $(X, \mathscr{C}_b(T, V), K(X))$ satisfies property* (R_1).

Proof. Suppose $g \in \mathscr{C}_b(T, V)$, $F \in K(X)$, $r_1 > 0$, and $r_2 > 0$ are such that

$$\mathscr{C}_b(T, V) \cap \left(\bigcap \{ \bar{B}(f; r_2) : f \in F \} \right) \neq \phi \qquad \text{and} \qquad r(F; g) < r_1 + r_2.$$

Define Φ: $T \to CC(V)$ with values

$$\Phi(t) := V \cap \bar{B}(g(t); r_1) \cap \left(\bigcap \{ \bar{B}(f(t); r_2) : f \in F \} \right).$$

By Proposition 2.3, $(U, V, K(U))$ satisfies property (R_1) and since $F(t) \in K(U)$, we have $\Phi(t) \neq \phi$ for each $t \in T$. We assert that Φ is l.s.c. Indeed, suppose $\Phi(t_0) \cap B(u; \alpha) \neq \phi$ for some $u \in V$. Pick up $v \in \phi(t_0) \cap B(u; \alpha)$ and $\beta > 0$ such that $\|v - u\| < \beta < \alpha$. Let $\varepsilon = \alpha - \beta$. Then we have $\|v - g(t_0)\| \leqslant r_1$ and $F(t_0) \subset \bar{B}(v; r_2)$. By upper semicontinuity of $t \to F(t)$, there is a neighbourhood N_1 of t_0 such that $F(t) \subset B(v; r_2 + \varepsilon)$ for every $t \in N_1$ and by continuity of g, there is a neighbourhood N_2 of t_0 such that $\|g(t) - g(t_0)\| < \varepsilon$ for all $t \in N_2$. Let $N = N_1 \cap N_2$. Then $\bar{B}(v; \varepsilon) \cap \bar{B}(f(t); r_2) \neq \phi$ and $\bar{B}(v; \varepsilon) \cap \bar{B}(g(t); r_1) \neq \phi$ for each $f \in F$ and $t \in N$. Since U is a Lindenstrauss space, $F(t) \cup \{g(t)\} \in K(U)$ and V is an M-ideal, we have by [16, Lemma 2.1] $\Phi(t) \cap \bar{B}(v; \varepsilon) \neq \phi$ for each $t \in N$. Therefore, $\Phi(t) \cap B(u; \alpha) \neq \phi$ for each $t \in N$ and this proves that Φ is l.s.c. By Michael's selection theorem, Φ has a continuous selection h. Clearly $h \in \mathscr{C}_b(T, V) \cap \bar{B}(g; r_1) \cap S_{r_2}(F)$. ∎

PROPOSITION 2.9. *If T is an arbitrary set, $X = l_\infty(T, \mathscr{R})$, and V is a closed linear subspace of X with the property that for each $g \in V$ and $k > 0$, the function $(g \wedge k) \vee (-k)$ belongs to V, then $(X, V, CB(X))$ satisfies property* (R_1).

Proof. By a translation, it would suffice to prove that if $F \in CB(X)$ and $r_1 > 0$ and $r_2 > 0$ are such that $V \cap S_{r_2}(F) \neq \phi$ and $r(F; \theta) < r_1 + r_2$, then $V \cap \bar{B}(\theta; r_1) \cap S_{r_2}(F) \neq \phi$. Pick $g \in V \cap S_{r_2}(F)$ and let $h = (g \wedge r_1) \vee (-r_1)$. Then $h \in V \cap \bar{B}(\theta; r_1)$. We show that $r(F; h) \leqslant r_2$. Let $t \in T$ be given. If $|g(t)| \leqslant r_1$, then $h(t) = g(t)$ and therefore $|f(t) - h(t)| = |f(t) - g(t)| \leqslant \|f - g\| \leqslant r_2$, for each $f \in F$. If $g(t) > r_1$, then $h(t) = r_1$ and we have

$$-r_2 \leqslant f(t) - g(t) < f(t) - r_1 = f(t) - h(t) < r_2, \qquad \text{for each} \quad f \in F.$$

Last, if $g(t) < -r_1$, then $h(t) = -r_1$, and in this case we have

$$-r_2 = -(r_1 + r_2) + r_1 < f(t) + r_1 = f(t) - h(t) < f(t) - g(t) \leqslant \|f - g\| \leqslant r_2,$$

for each $f \in F$. Thus $|f(t) - h(t)| \leqslant r_2$ for each $t \in T$ and $f \in F$, which gives $r(F; h) \leqslant r_2$. ∎

COROLLARY 2.10. *If T is a compact Hausdorff space, $X = \mathscr{C}(T, \mathscr{R})$, and V is a closed subalgebra of X containing nonzero constants, then $(X, V, CB(X))$ satisfies property* (R_1).

Proof. This follows from the well known fact that any closed subalgebra of $\mathscr{C}(T, \mathscr{R})$ is a sublattice. Indeed, if V contains nonzero constants, then the condition of the preceding proposition is satisfied. ∎

3. PROPERTIES (R_2), (R_3), AND (R_4)

Given a family $\mathscr{V} \subset CL(X)$ and a family $\mathscr{F} \subset CB(X)$, where X is a normed space, we introduce properties (R_2), (R_3), (R_4), and $(\tilde{R}_4)$ given by the following definitions.

DEFINITION 3.1. Let $V \in \mathscr{V}$ be given. The triplet $(X, V, \mathscr{F})$ is said to satisfy *property* (R_2) if, for every $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that given $F \in \mathscr{F}$ with $\operatorname{rad}_V(F) \leqslant r$ and $u \in V$ such that $r(F; u) < r + \delta$, there exists $v \in V$ such that $\|u - v\| < \varepsilon$ and $r(F; v) \leqslant r$, or equivalently, $\operatorname{dist}(u; S_r(F) \cap V) < \varepsilon$, whenever $F \in \mathscr{F}$ with $\operatorname{rad}_V(F) \leqslant r$ and $u \in (\bigcap_{y \in F} \{B(y; r + \delta)\}) \cap V$.

DEFINITION 3.2. Given $V \in \mathscr{V}$, the triplet $(X, V, \mathscr{F})$ is said to satisfy *property* (R_3) if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $F \in \mathscr{F}$, every $r \geqslant \operatorname{rad}_V(F)$ and each $u \in V$ satisfying $r(F; u) < r + \delta$, there exists $v \in V$ such that $\|u - v\| < \varepsilon$ and $r(F; v) \leqslant r$.

DEFINITION 3.3. The triplet $(X, \mathscr{V}, \mathscr{F})$ is said to satisfy *property* (R_4) if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $V \in \mathscr{V}$, $F \in \mathscr{F}$, and $r \geqslant \operatorname{rad}_V(F)$, for each $u \in V$ satisfying $r(F; u) < r + \delta$, there exists $v \in V$ such that $\|u - v\| < \varepsilon$ and $r(F; v) \leqslant r$. The triplet $(X, \mathscr{V}, \mathscr{F})$ is said to satisfy property $(\tilde{R}_4)$ if, given $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that for every $V \in \mathscr{V}$ and $F \in \mathscr{F}$ such that $r \geqslant \operatorname{rad}_V(F)$ and each $u \in V$ satisfying $r(F; u) < r + \delta$, $\operatorname{dist}(u; S_r(F) \cap V) < \varepsilon$.

Clearly, if $(X, \mathscr{V}, \mathscr{F})$ satisfies property (R_4), then it satisfies property $(\tilde{R}_4)$ and $(X, V, \mathscr{F})$ satisfies (R_3) for every $V \in \mathscr{V}$.

We remark that property (R_2) is stronger than property (P_1) of Mach [17], which is obtained by taking $r = \operatorname{rad}_V(F)$ for each $F \in \mathscr{F}$ in the statement of (R_2). Likewise, property (R_3) is stronger than property (P_2) of [17], which is obtained by taking $r = \operatorname{rad}_V(F)$ for each $F \in \mathscr{F}$ in the statement of (R_2). We also observe that (R_2) reduces to property (A) (resp. is stronger than property (A)) of [20] when $\mathscr{F}$ consists of all singletons (resp. contains all singletons) in X. It is clear that if the triplet $(X, V, \mathscr{F})$ satisfies (R_2), then V satisfies r.c.p. for $\mathscr{F}$.

The next four propositions give examples of triplets $(X, V, \mathscr{F})$ satisfying property (R_2). Recall [22, p. 368] that a set $V \in CL(X)$ is said to be boundedly compact if $V \cap \bar{B}(x; r)$ is compact for every $x \in X$ and $r > 0$.

PROPOSITION 3.4. *Suppose V is a boundedly compact subset of X; then $(X, V, CB(X))$ satisfies property* (R_2).

Proof. Assume the contrary. Then there are numbers $\varepsilon > 0$ and $r > 0$ and a set $F \in \mathscr{F}$ with $\text{rad}_V(F) \leqslant r$, such that for each n, there exists $v_n \in V$ such that $r(F; v_n) < r + 1/n$ and $\text{dist}(v_n; S_r(F) \cap V) \geqslant \varepsilon$. Clearly $\{v_n\} \subset S_{r+1}(F) \subset \bar{B}(\theta; \text{diameter}(F) + r + 1) \cap V$, which is compact. Therefore $\{v_n\}$ has a convergent subsequence $\{v_{n_k}\}$ converging to v_0 in V. Then $v_0 \in S_r(F) \cap V$; but $\text{dist}(v_0; S_r(F) \cap V) \geqslant \varepsilon$, which is a contradiction. ▮

PROPOSITION 3.5. *If $X = l_1$ and V is a w^*-closed convex subset of X, then $(X, V, CB(X))$ satisfies property* (R_2).

Proof. Assume the contrary. Then there are numbers $\varepsilon > 0$ and $r > 0$ and a set $F \in \mathscr{F}$ with $\text{rad}_V(F) \leqslant r$, such that for each n, there exists $v_n \in V$ such that $r(F; v_n) < r + 1/n$ and $\text{dist}(v_n; S_r(F) \cap V) \geqslant \varepsilon$. The proof of Proposition 2 of [17] is now easily seen to work here with $\text{rad}_V(F)$ replaced by r. ▮

Recall [6] that X is said to be *quasi uniformly convex with respect to a set $V \in CC(X)$* if for every $0 < \varepsilon < 1$ there exists $0 < \tilde{\delta} = \tilde{\delta}(\varepsilon) \leqslant \varepsilon$ such that given u, v in V, there exists $u_0 \in V$ with $\|u - u_0\| \leqslant \varepsilon$ and such that $\bar{B}(u; 1) \cap \bar{B}(v; 1 - \tilde{\delta}) \subset \bar{B}(u_0; 1 - \tilde{\delta})$. In this case, we say that the pair (X, V) satisfies property (QUC). If V is a closed linear subspace, then by a translation, we may assume $v_0 = \theta$ in the above definition. We also recall (cf. [6]) that in case V is a closed linear subspace of X, then X is said to be *uniformly convex with respect to V*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $u - v \in V$, $\|u\| = \|v\| = 1$, $\|u - v\| \geqslant \varepsilon$ imply $\|\frac{1}{2}(u + v)\| \leqslant 1 - \delta$. It is known [4, Proposition 2.2] that X is uniformly convex with respect to V if and only if (X, V) satisfies (QUC) and we can take u_0 in the line segment connecting u and v in the definition of (QUC).

PROPOSITION 3.6. *If V is a closed linear subspace of a Banach space X and (X, V) satisfies property* (QUC), *then $(X, V, CB(X))$ satisfies property* (R_2).

Proof. Let $\varepsilon>0$ and $r>0$ be given. By scaling we may assume, without loss of generality, that $r=1$ and $\varepsilon<1$. Take $0<\tilde{\varepsilon}<\varepsilon/(1+\varepsilon)$ and let $\mu=\tilde{\delta}(\tilde{\varepsilon})\leqslant\tilde{\varepsilon}$ as in the definition of property (QUC). Take $\delta=\mu/(1-\mu)$. Let $F\in CB(X)$ be such that $\text{rad}_V(F)\leqslant 1$ and pick $u\in V$ such that $r(F;u)<1+\delta$. Then $r(F/(1+\delta);u/(1+\delta))<1$. By [4, Proposition 2.4(a)], $\text{Cent}_V(F)\neq\phi$. Therefore, we can pick up $v\in V$ such that $r(F;v)\leqslant 1$. Then $r(F/(1+\delta);v/(1+\delta))\leqslant 1-\mu$ and

$$\frac{F}{(1+\delta)}\subset\bar{B}\left(\frac{u}{(1+\delta)};1\right)\cap\bar{B}\left(\frac{v}{(1+\delta)};1-\mu\right)\subset\bar{B}(u_0;1-\mu)$$

for some $u_0\in V$ with $\|u_0-u/(1+\delta)\|\leqslant\tilde{\varepsilon}$ by property (QUC). Let $\tilde{u}=(1+\delta)u_0$. Then $\tilde{u}\in V$, $\|\tilde{u}-u\|\leqslant\tilde{\varepsilon}(1+\delta)=\tilde{\varepsilon}/(1-\mu)\leqslant\tilde{\varepsilon}/(1-\tilde{\varepsilon})<\varepsilon$, and $r(F;\tilde{u})\leqslant 1$. ∎

PROPOSITION 3.7. *If X is locally uniformly convex, $V\in CC(X)$, and V has property r.c.p. for $K(X)$, then $(X, V, K(X))$ satisfies property* (R_2).

Proof. This is an easy modification of the proof of Proposition 4 of [17]. ∎

Given a set $V\in CL(X)$, recall [15] that the pair (X, V) is said to satisfy property (P) (called (P_1) in [15]) if given $\varepsilon>0$ and $r>0$, there exist $\delta>0$ and a function $h: V\times V\to V$ such that for every θ, with $|\theta|<\delta$, we have $h(x,y)\in\bar{B}(x;\varepsilon)$ and $\bar{B}(x;r+\delta)\cap\bar{B}(y;r+\theta)\subset\bar{B}(h(x,y);r+\theta)$. The pair (X, V) is said to satisfy $(\tilde{P})$ (called (P_2) in [15]) if it satisfies (P) with h continuous. It is shown in [15] that for a Banach space X if (X, V) satisfies (P), then V satisfies r.c.p. for $CB(X)$. Also if X is uniformly convex Banach space and $V\in CC(X)$, then (X, V) satisfies $(\tilde{P})$ and, moreover, if (X, V) satisfies $(\tilde{P})$, then $(l_\infty(T,X),\mathscr{C}_b(T,V))$ satisfies (P) for any topological space T.

PROPOSITION 3.8. *If X is a Banach space, $V\in CL(X)$, and the pair (X, V) satisfies property* (P), *then $(X, V, CB(X))$ satisfies property* (R_2).

Proof. Let $\varepsilon>0$ and $r>0$ be given. Let $F\in CB(X)$ be such that $\text{rad}_V(F)\leqslant r$. Since (X, V) satisfies (P), there exists $\delta>0$ such that for each u, w in V, there is an element $v\in V$ such that $\bar{B}(u;r+\delta)\cap\bar{B}(w;r+\theta)\subset\bar{B}(v;r+\theta)$ for every θ, with $|\theta|<\delta$. Let $u\in V$ be such that $r(F;u)<r+\delta$. By [15, Theorem 2] we can pick up $w\in\text{Cent}_V(F)$. Then there is an element $v\in V$ such that $\|u-v\|\leqslant\varepsilon$ and $F\subset\bar{B}(u;r+\delta)\cap\bar{B}(w;r)\subset\bar{B}(v;r)$. Therefore,

$$v\in V\cap S_r(F)\qquad\text{and}\qquad\text{dist}(u;V\cap S_r(F))\leqslant\varepsilon.\ ∎$$

The relationship between the properties (R_1), (R_2), and (R_3) is clarified in the next proposition.

PROPOSITION 3.9. *If X is a normed space, $V \in CL(X)$, and $\mathscr{F} \subset CB(X)$, then for the triplet $(X, V, \mathscr{F})$ we have*

$$(R_1) \Rightarrow (R_3) \Rightarrow (R_2).$$

Proof. $(R_1) \Rightarrow (R_3)$: Suppose $(X, V, \mathscr{F})$ satisfies (R_1). Let $\varepsilon > 0$ be given and let δ be any number such that $0 < \delta < \varepsilon$. Let $F \in \mathscr{F}$ and $r \geqslant \operatorname{rad}_V(F)$. Let $u \in V$ be such that $r(F; u) < r + \delta$. By Theorem 2.2, $S_r(F) \cap V \neq \phi$. Hence by (R_1), $V \cap \bar{B}(u; \delta) \cap S_r(F) \neq \phi$. Therefore $\operatorname{dist}(u; S_r(F) \cap V) \leqslant \delta < \varepsilon$, and this shows that $(X, V, \mathscr{F})$ satisfies (R_2).

$(R_3) \Rightarrow (R_2)$: This is obvious. ∎

Remarks 3.10. (i) If $X = l_1$ and $V = X$, then by Proposition 3.5, the triplet $(X, V, CB(X))$ satisfies property (R_2); but it is known [4, Corollary 2.7] that (X, V) does not satisfy property (QUC).

(ii) Let X be the M-space $\{f \in \mathscr{C}[0, 1] : f(1/2n) = (1/n) f(1/(2n-1)), n = 1, 2, ...\}$ and $V = X$. Then X is a Lindenstrauss space and by Proposition 2.3, $(X, V, K(X))$ satisfies property (R_1); but it is known from [4, Example 4.7] that (X, V) does not satisfy (QUC).

PROPOSITION 3.11. *If for each $V \in \mathscr{V}$, $(X, V, \mathscr{F})$ satisfies property* (R_1), *then $(X, \mathscr{V}, \mathscr{F})$ satisfies property* (R_4).

Proof. Let $\varepsilon > 0$ be given. From the proof of $(R_1) \Rightarrow (R_3)$ in the previous proposition, it is clear that for any δ, $0 < \delta < \varepsilon$, for every $V \in \mathscr{V}$, $F \in \mathscr{F}$, and $r \geqslant \operatorname{rad}_V(F)$, if $u \in V$ is such that $r(F; u) < r + \delta$, then $\operatorname{dist}(u; S_r(F) \cap V) < \varepsilon$. Thus $(X, \mathscr{V}, \mathscr{F})$ satisfies (R_4). ∎

PROPOSITION 3.12. *If X is a uniformly convex Banach space, then the triplet $(X, CC(X), CB(X))$ satisfies property* ($\tilde{R}_4$).

Proof. Let $\varepsilon > 0$ and $r > 0$ be given. Let $V \in CC(X)$ and $F \in CB(X)$ be such that $\operatorname{rad}_V(F) \leqslant r$. By [15, Proposition 1], there exists $\delta > 0$ such that for every $x, y \in X$ and every θ with $|\theta| < \delta$, we have

$$(*) \quad \bar{B}(x; r + \delta) \cap \bar{B}(y; r + \theta) \subset \bar{B}(\psi_\varepsilon(x, y); r + \theta),$$

where

$$\psi_\varepsilon(x, y) = \begin{cases} y, & \text{if } \|x - y\| \leqslant \varepsilon \\ (1 - \varepsilon \|x - y\|^{-1})x + \varepsilon \|x - y\|^{-1} y, & \text{if } \|x - y\| > \varepsilon. \end{cases}$$

Now let $u \in V$ be such that $r(F; u) < r + \delta$ and pick up $v \in \mathrm{Cent}_V(F)$. Then $F \subset \bar{B}(u; r+\delta) \cap \bar{B}(v; r) \subset \bar{B}(\psi_\varepsilon(u, v); r)$. Since $\psi_\varepsilon(u, v) \in V$ and $\|u - \psi_\varepsilon(u, v)\| \leqslant \varepsilon$, we conclude that $\mathrm{dist}(u, S_r(F) \cap V) \leqslant \varepsilon$. ∎

PROPOSITION 3.13. *If T is a topological space, U is a uniformly convex Banach space, $X = l_\infty(T, U)$, and $\mathscr{V} := \{V : V$ is a closed convex $l_\infty(T, [0, 1])$-submodule of $l_\infty(T, U)\}$, then $(X, \mathscr{V}, CB(X))$ satisfies property $(\tilde{\mathrm{R}}_4)$.*

Proof. Let $\varepsilon > 0$, $r > 0$ be given and let $\delta > 0$ be such that (*) holds for all $x, y \in U$. Let $V \in \mathscr{V}$ and $F \in CB(X)$ be such that $\mathrm{rad}_V(F) \leqslant r$. Then for f, g in $l_\infty(T, U)$, $\bar{B}(f; r+\delta) \cap \bar{B}(g, r+\theta) \subset \bar{B}(h_\varepsilon(f, g); r+\theta)$ for every θ with $|\theta| < \delta$, where $h_\varepsilon(f, g)(t) = \psi_\varepsilon(f(t), g(t))$, with ψ_ε as in the previous proposition. Let $f \in V$ be such that $r(F; f) < r + \delta$. By [21, Corollary 2.3], we can pick up $g \in \mathrm{Cent}_V(F)$. Then $F \subset \bar{B}(f; r+\delta) \cap \bar{B}(g; r) \subset \bar{B}(h_\varepsilon(f, g); r)$. Since V is a convex $l_\infty(T, [0, 1])$-submodule, $h_\varepsilon(f, g) \in V$ and $\|f - h_\varepsilon(f, g)\| \leqslant \varepsilon$. Therefore, $\mathrm{dist}(f; S_r(F) \cap V) \leqslant \varepsilon$. ∎

COROLLARY 3.14. *With T, U, and X as in the previous proposition, if $\mathscr{V} := \{V : V$ is a closed $\mathscr{C}_b(T, \mathscr{K})$-submodule of $\mathscr{C}_b(T, U)\}$, then $(X, \mathscr{V}, CB(X))$ satisfies property $(\tilde{\mathrm{R}}_4)$.*

4. CONTINUITY OF $\mathrm{Cent}_V(\cdot)$-MAP

As in the previous section, let X be a normed space and let the families $\mathscr{V} \subset CL(X)$ and $\mathscr{F} \subset CB(X)$ be given.

THEOREM 4.1. *Let $V \in \mathscr{V}$ and suppose $(X, V, \mathscr{F})$ satisfies property (R_2). Then the Cent_V-map: $F \to \mathrm{Cent}_V(F)$ of $\mathscr{F}$ equipped with τ_H into $CL(V)$ is u.H.s.c.*

Proof. Since $(X, V, \mathscr{F})$ satisfies property $(\mathrm{R}_2) \Rightarrow (V, \mathscr{F})$ satisfies property (P_1) of [17], this follows readily from [17, Theorem 5]. ∎

THEOREM 4.2. *Let $V \in \mathscr{V}$ and suppose $(X, V, \mathscr{F})$ satisfies property (R_3). Then the Cent_V-map: $F \to \mathrm{Cent}_V(F)$ of $\mathscr{F}$ equipped with τ_H into $CL(V)$ is uniformly H-continuous.*

Proof. Let $\varepsilon > 0$ be given and select $\delta > 0$ as in Property (R_3). Since $(X, V, \mathscr{F})$ satisfies property $(\mathrm{R}_3) \Rightarrow (V, \mathscr{F})$ satisfies property (P_2) in [17], it follows exactly as in the proofs of Theorems 5 and 6 of [17], that for every δ_1, $0 < \delta_1 < \delta$, $F, G \in \mathscr{F}$ and $H(F, G) < \delta_1/2$, imply $h(\mathrm{Cent}_V(G), \mathrm{Cent}_V(F)) \leqslant \varepsilon$ and $h(\mathrm{Cent}_V(F), \mathrm{Cent}_V(G)) \leqslant \varepsilon$, i.e., $H(\mathrm{Cent}_V(F), \mathrm{Cent}_V(G)) \leqslant \varepsilon$. ∎

Remarks 4.3. (i) If $V \in CC(X)$ and the triplet $(X, V, CB(X))$ satisfies property (R_3), then the pair (X, V) satisfies property (QUC). This follows immediately from [6, Theorem in 6.1].

(ii) By Remark 3.10(i), the triplet $(l_1, l_1, CB(l_1))$ satisfies (R_2) but not (R_3).

(iii) If the pair (X, V) satisfies property (P), then Cent_V-map: $F \to \text{Cent}_V(F)$ is H-continuous. This follows from [15, Theorem 3], Proposition 3.8 and Theorem 4.1.

THEOREM 4.4. *Let $V \in \mathscr{V}$ and suppose $(X, V, \mathscr{F})$ satisfies property* (R_1). *Then the* Cent_V-map: $F \to \text{Cent}_V(F)$ *of $\mathscr{F}$ equipped with τ_H into $CL(V)$ is Lipschitz H-continuous. In fact*

$$H(\text{Cent}_V(F), \text{Cent}_V(G)) \leqslant 2H(F, G)$$

for all F, G in $\mathscr{F}$ and the constant 2 is, in general, the best constant.

Proof. By Proposition 3.9, $(X, V, \mathscr{F})$ satisfies $(R_1) \Rightarrow (X, V, \mathscr{F})$ satisfies (R_3). From the proof of Proposition 3.9, it is clear that given $\varepsilon > 0$, any number δ, $0 < \delta < \varepsilon$, works in the definition of (R_3), when $(X, V, \mathscr{F})$ satisfies (R_1). From the proof of the preceding theorem, it follows that F, G in $\mathscr{F}$ and $H(F, G) < \varepsilon/2$ imply $H(\text{Cent}_V(F), \text{Cent}_V(G)) \leqslant \varepsilon$. Therefore, $H(\text{Cent}_V(F), \text{Cent}_V(G)) \leqslant 2H(F, G)$, for all F, G in $\mathscr{F}$. To show that this inequality is sharp, let $X = \mathscr{R}^3$ equipped with the box norm, let V be the one dimensional space spanned by $(1, 1, 0)$, and let $\mathscr{F}$ be the singletons in X. It is easy to see that $(X, V, \mathscr{F})$ satisfies (R_1). Let $F = \{(0, 0, 3)\}$ and $G = \{(1, -1, 2)\}$. It is easily seen that $\text{Cent}_V(F) = \{(\lambda, \lambda, 0) : |\lambda| \leqslant 3\}$, $\text{Cent}_V(G) = \{(\lambda, \lambda, 0) : |\lambda| \leqslant 1\}$, $H(F, G) = 1$ and $H(\text{Cent}_V(F), \text{Cent}_V(G)) = 2$, which shows that 2 is the best constant. ∎

THEOREM 4.5. *If the triplet $(X, \mathscr{V}, \mathscr{F})$ satisfies property* (R_4), *then the family of set-valued maps* $\{\text{Cent}_V(\cdot): V \in \mathscr{V}\}$ *is uniformly equi-H-continuous on $\mathscr{F}$ equipped with τ_H.*

Proof. Let $\varepsilon > 0$ be given and let $\delta > 0$ be as in the definition of Property (R_4). It follows exactly as in the proof of Theorem 4.2 that for every $V \in \mathscr{V}$, $F, G \in \mathscr{F}$ and $H(F, G) < \delta/2$ imply $H(\text{Cent}_V(F), \text{Cent}_V(G)) \leqslant \varepsilon$. ∎

THEOREM 4.6. *If the triplet $(X, \mathscr{V}, \mathscr{F})$ satisfies property* $(\tilde{R}_4)$, *then the family of set-valued maps* $\{\text{Cent}_V(\cdot): V \in \mathscr{V}\}$ *is equi-u.H.s.c. on $\mathscr{F}$ equipped with τ_H: given $F_0 \in \mathscr{F}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that* $h(\text{Cent}_V(F), \text{Cent}_V(F_0)) < \varepsilon$, *for every $V \in \mathscr{V}$, whenever $F \in \mathscr{F}$ and $H(F, F_0) < \delta$.*

Proof. This follows exactly on the same lines as the proof of Theorem 4.1 using the definition of Property ($\tilde{R}_4$). ∎

In conjunction with Proposition 3.10, Theorem 2.2, and the previous propositions and corollaries giving examples satisfying property (R_1) we obtain:

COROLLARY 4.7. *Each $V \in \mathscr{V}$ satisfies r.c.p. for $\mathscr{F}$ and the family $\{\mathrm{Cent}_V(\cdot): V \in \mathscr{V}\}$ is equi-Lipschitz H-continuous on $\mathscr{F}$ equipped with τ_H, with Lipschitz constant 2, in each of the following cases*:

(1) *X is a Lindenstrauss space, $\mathscr{V}$ = family of all M-ideals in X, and $\mathscr{F} = K(X)$;*

(2) *T is a compact Hausdorff space, U a Lindenstrauss space, $X = \mathscr{C}(T, U)$, $\mathscr{V}$ = the family of all* W–S *subspaces in X, and $\mathscr{F} = K(X)$;*

(3) *T is a compact Hausdorff space, $X = \mathscr{C}(T, \mathscr{R})$, $\mathscr{V}$ = the family of all closed subalgebras in X, and $\mathscr{F} = K(X)$;*

(4) *T is a compact Hausdorff space, U a Lindenstrauss space, $X = \mathscr{C}(T, U)$, $\mathscr{V} = \{V_E : E$ a closed subset of $T\}$, where $V_E := \{g \in \mathscr{C}(T, U) : g|_E = \theta\}$ and $\mathscr{F} = K(X)$;*

(5) *T is a paracompact Hausdorff space, U a Lindenstrauss space, $X = l_\infty(T, U)$, $\mathscr{V} = \{\mathscr{C}_b(T, M) : M$ is an M-ideal in $U\}$, and $\mathscr{F} = K(X)$;*

(6) *T is an arbitrary set, $X = l_\infty(T, \mathscr{R})$, $\mathscr{V} = \{V : V$ is a closed linear subspace of X satisfying the condition in Proposition 2.9$\}$, and $\mathscr{F} = CB(X)$;*

(7) *T is a compact Hausdorff space, $X = \mathscr{C}(T, \mathscr{R})$, $\mathscr{V}$ = the family of all closed subalgebras of X containing nonzero constants, and $\mathscr{F} = CB(X)$.*

Some of the cases in the preceding corollary are improvements of some of the known results: (1) improves Proposition 3 and Corollary 7(v) of [17]; (2) improves [7, Corollary 3.19] and [25, Theorem 2.1]; (3) and (7) are improvements of [25, Corollary 2.3] and partial improvements of [24, Theorem 1]; (6) improves [20, Example 5].

5. RESTRICTED CENTER SELECTION

Let T be a topological space, let U be a Banach space, and let $X = l_\infty(T, U)$. Suppose a set $V \in CL(U)$ and a family $\mathscr{F} \subset CB(X)$ are given. By an abuse of notation, we continue to denote by $\mathscr{C}_b(T, V)$, the closed subset $\{f \in \mathscr{C}_b(T, U) : f(T) \subset V\}$ of $l_\infty(T, U)$, which is convex if V is convex. In this section, we mainly address the following questions:

(1) When does $\mathscr{C}_b(T, V)$ satisfy r.c.p. for $\mathscr{F}$?

(2) When does $\mathscr{C}_b(T, V)$ have a continuous restricted center selection on $\mathscr{F}$?

In case $\mathscr{F} = K(X)$, answers to both these questions are provided in the next theorem.

THEOREM 5.1. *Suppose V admits a continuous restricted center selection on $K(U)$. Then $\mathscr{C}_b(T, V)$ satisfies r.c.p. for $K(X)$, where $X = \mathscr{C}_b(T, U)$. Moreover, if either*

(i) *T is compact Hausdorff*

or

(ii) *V has a continuous restricted center selection on $K(U)$ which is uniformly continuous on the sets $\{A \in K(U) : \operatorname{rad}_V(A) \leqslant r\}$ equipped with τ_H for each $r > 0$,*

then $\mathscr{C}_b(T, V)$ has a continuous restricted center selection on $K(X)$.

Proof. Let $F \in K(X)$ and let $A \to C(A) \in \operatorname{Cent}_V(A)$ be a continuous restricted center selection for V on $K(U)$. By Lemma 1.1 and hypothesis, $t \to C(F(t))$ is H-continuous. Define the function $\tilde{C}: K(X) \to \mathscr{C}(T, V)$ by $\tilde{C}(F)(t) = C(F(t))$ for $F \in K(X)$ and $t \in T$. It is easily verified that for each $t \in T$, $\operatorname{rad}_V(F(t)) \leqslant \operatorname{rad}_{\mathscr{C}_b(T, V)}(F)$. Therefore $\|\tilde{C}(F)(t)\| \leqslant \|f\| + \operatorname{rad}_{\mathscr{C}_b(T, V)}(F)$ for any $f \in F$ and each $t \in T$, whence we conclude that $\tilde{C}$ maps $K(X)$ into $\mathscr{C}_b(T, V)$. We assert that $\tilde{C}(F) \in \operatorname{Cent}_{\mathscr{C}_b(T, V)}(F)$ for each $F \in K(X)$. Indeed, we have

$$\begin{aligned} \operatorname{rad}_{\mathscr{C}_b(T, V)}(F) &\leqslant \sup_{f \in F} \|\tilde{C}(F) - f\| = \sup_{f \in F} \sup_{t \in T} \|\tilde{C}(F)(t) - f(t)\| \\ &= \sup_{t \in T} r(F(t); \tilde{C}(F)(t)) = \sup_{t \in T} r(F(t); C(F(t))) \\ &= \sup_{t \in T} \operatorname{rad}_V(F(t)) \leqslant \operatorname{rad}_{\mathscr{C}_b(T, V)}(F). \end{aligned}$$

Therefore $r(F; \tilde{C}(F)) = \operatorname{rad}_{\mathscr{C}_b(T, V)}(F)$, and this proves that $\mathscr{C}_b(T, V)$ satisfies r.c.p. for $K(X)$. We assert that $F \to \tilde{C}(F)$ is a continuous map of $K(X)$ equipped with τ_H into $\mathscr{C}_b(T, V)$ under either of the two assumptions (i) or (ii). First suppose (i) is satisfied and, assume the contrary, that $F \to \tilde{C}(F)$ is not continuous. Then there are a net $\langle F_\lambda \rangle$ in $K(X)$, τ_H-convergent to F_0 in $K(X)$, and a number $\varepsilon > 0$, such that

$$(*) \quad \|\tilde{C}(F_\lambda) - \tilde{C}(F_0)\| \geqslant \varepsilon \qquad \text{for all } \lambda.$$

Pick $t_\lambda \in T$ for each λ such that $\|\tilde{C}(F_\lambda) - \tilde{C}(F_0)\| = \|\tilde{C}(F_\lambda)(t_\lambda) - \tilde{C}(F_0)(t_\lambda)\| = \|\tilde{C}(F_\lambda(t_\lambda)) - \tilde{C}(F_0(t_\lambda))\|$. Since T is compact, the net $\langle t_\lambda \rangle$ has a subnet $\langle t_\mu \rangle$ convergent to $t_0 \in T$.

Since $H(F_\mu(t_\mu), F_0(t_0)) \leqslant H(F_\mu(t_\mu), F_0(t_\mu)) + H(F_0(t_\mu), F_0(t_0)) \leqslant H(F_\mu, F_0) + H(F_0(t_\mu), F_0(t_0))$, by Lemma 1.1, $H-\lim_\mu F_\mu(t_\mu) = F_0(t_0)$, which contradicts (*) and establishes continuity of the map $F \to \tilde{C}(F)$. Next suppose (ii) is satisfied. Fix $F \in K(X)$. Let $\varepsilon > 0$ be given and let $\mathscr{A} := \{A \in K(U) : H(A, F(t)) < \varepsilon \text{ for some } t \in T\}$. Since $|\mathrm{rad}_V(A) - \mathrm{rad}_V(F(t))| \leqslant H(A, F(t))$, we have

$$\mathscr{A} \subset \{A \in K(U) : \mathrm{rad}_V(A) < \mathrm{rad}_{\mathscr{C}_b(T,V)}(F) + \varepsilon\}$$

and by hypothesis, there is a $\delta > 0$ such that for A, B in $\mathscr{A}$ with $H(A, B) < \delta$, we have $\|C(A) - C(B)\| < \varepsilon$. We may assume $0 < \delta < \varepsilon$. Now let $G \in K(X)$ be such that $H(F, G) < \delta$. Then $H(F(t), G(t)) < \delta$ and since $F(t), G(t) \in \mathscr{A}$, we have $\|C(F(t)) - C(G(t))\| < \varepsilon$ for each $t \in T$. Therefore $\|\tilde{C}(F) - \tilde{C}(G)\| < \varepsilon$ and we conclude that $\tilde{C}$ is continuous at F. ∎

The preceding theorem extends [13, Theorem 2.1] as well as [20, Theorem 1]. In conjunction with [4, Proposition 2.4(a)], Remark 4.3(iii), and Theorem 4.1 we obtain:

COROLLARY 5.2. *If T is an arbitrary topological space, then $\mathscr{C}_b(T, V)$ has r.c.p. for $K(X)$, where $X = \mathscr{C}_b(T, U)$ and, moreover, $\mathscr{C}_b(T, V)$ has a continuous restricted center selection on $K(X)$ in case T is compact Hausdorff, in each of the following cases*:

(1) *V is a closed linear subspace of U and the pair (U, V) satisfies property* (QUC);

(2) *$V \in CL(U)$ and the pair (U, V) satisfies property* (P);

(3) *$V \in CL(U)$ and the triplet $(U, V, K(U))$ satisfies property* (R_1).

We remark that in the previous corollary (1) improves [20, Corollary 4(g)] and (2) is a partial improvement of [15, Corollary 5].

In conjunction with Proposition 2.4, Corollary 2.7, and Proposition 3.9 the preceding theorem gives:

COROLLARY 5.3. *Let S, T be compact Hausdorff spaces, U a Lindenstrauss space and let $X = \mathscr{C}(S \times T, U)$; then $\mathscr{C}(S, V)$ has a continuous restricted centre selection on $K(X)$ in each of the following cases*:

(1) *V is a* W–S *subspace of $\mathscr{C}(T, U)$*;

(2) *$V = \{g \in \mathscr{C}(T, U) : g|_E = \theta$, for a given closed $E \subset T\}$.*

Proof. We need only identify the Banach spaces $\mathscr{C}(S, \mathscr{C}(T, U))$, and $\mathscr{C}(S \times T, U)$. ∎

The following Corollary which is obtained using Proposition 2.3, Proposition 2.8, and Proposition 3.9 along with the preceding theorem is also of independent interest.

COROLLARY 5.4. *If T is an arbitrary topological space, U is a Lindenstrauss space, and V is an M-ideal in U, then $\mathscr{C}_b(T, V)$ has r.c.p. for $K(X)$, where $X=\mathscr{C}_b(T, U)$, and in case T is paracompact Hausdorff, $\mathscr{C}_b(T, V)$ has a continuous restricted center selection on $K(Y)$, where $Y=l_\infty(T, U)$. In particular, $\mathscr{C}_b(T, U)$ admits centers for $K(X)$ if T is arbitrary, and $\mathscr{C}_b(T, U)$ has a continuous center selection on $K(Y)$ if T is paracompact Hausdorff.*

Theorem 5.2 in conjunction with [6, Theorem in 4.1 and Theorem 6.1] also yields:

COROLLARY 5.5. *If T is an arbitrary topological space, U is an uniformly convex Banach space and $V \in CC(U)$, then $\mathscr{C}_b(T, V)$ has a continuous restricted center selection on $K(X)$, where $X=\mathscr{C}_b(T, U)$.*

We remark that the preceding corollary is also a consequence of Remark 4.3(iii) and [15, Theorem 3].

THEOREM 5.6. *Let T be a paracompact Hausdorff space, U be a Banach space, and $X=\mathscr{C}_b(T, U)$. Let $V \in CC(U)$ and $\mathscr{F} \subset CB(X)$ be such that for each $F \in \mathscr{F}$, the set-valued map $t \to \overline{F(t)}$ of T into $CB(U)$ is u.H.s.c. If the triplet $(U, V, CB(U))$ satisfies property (R_2), then $\mathscr{C}_b(T, V)$ has r.c.p. for $\mathscr{F}$.*

Proof. Let $F \in \mathscr{F}$ be given. For each $t \in T$, define

$$\Phi(t) := \{u \in V : r(\overline{F(t)}; u) \leqslant \mathrm{rad}_{\mathscr{C}_b(T,V)}(F)\}.$$

Since $\mathrm{rad}_V(\overline{F(t)}) \leqslant \mathrm{rad}_{\mathscr{C}_b(T,V)}(F)$, by property (R_2), $\Phi(t) \neq \phi$; also it is closed and convex. Thus Φ maps T into $CC(U)$. We claim that Φ is l.s.c. To this end, let $t_0 \in T$ and suppose $\Phi(t_0) \cap B(u_0; \alpha) \neq \phi$. Pick up $v \in \Phi(t_0)$ such that $\|v-u_0\| < \beta < \alpha$. Let $\varepsilon = \alpha - \beta$, $r = \mathrm{rad}_{\mathscr{C}_b(T,V)}(F)$ and choose $\delta > 0$ as in property (R_2). Since $t \to \overline{F(t)}$ is u.H.s.c. at t_0, there exists a neighbourhood N_{t_0} of t_0 such that for each $t \in N_{t_0}$, $h(\overline{F(t)}, \overline{F(t_0)}) < \delta$. Since $r(\overline{F(t)}; v) \leqslant r(\overline{F(t_0)}; v) + h(\overline{F(t)}, \overline{F(t_0)})$, we have $r(\overline{F(t)}; v) < r + \delta$, for each $t \in N_{t_0}$. Again since $\mathrm{rad}_V(\overline{F(t)}) \leqslant r$, by property (R_2), there exists $w_t \in V$ for each $t \in N_{t_0}$, such that $r(\overline{F(t)}; w_t) \leqslant r$ and $\|w_t - v\| < \varepsilon$. Since $\|w_t - u_0\| < \varepsilon + \beta = \alpha$, we have $w_t \in \Phi(t) \cap B(u_0; \alpha)$. Thus $\Phi(t) \cap B(u_0; \alpha) \neq \Phi$ for each $t \in N_{t_0}$, and this proves that Φ is l.s.c. By Michael's selection theorem [18], Φ has a continuous selection h. Since $r(\overline{F(t)}; h(t)) \leqslant r$, we have $\|h(t)\| \leqslant \|f\|_\infty + r$ for each $t \in T$ for any $f \in F$. Hence $h \in \mathscr{C}_b(T, V)$. Since $r(F; h) = \sup_{t \in T} r(F(t); h(t)) = \sup_{t \in T} r(\overline{F(t)}; h(t))$, we have $r(F; h) = r$ and $h \in \mathrm{Cent}_{\mathscr{C}_b(T,V)}(F)$. Thus $\mathscr{C}_b(T, V)$ has r.c.p. for $\mathscr{F}$. ∎

Remark 5.7. In view of Lemma 1.1, the preceding theorem holds for $\mathscr{F} = K(X)$. The preceding theorem in conjunction with Proposition 3.4, Proposition 3.5, and Proposition 3.7 yields:

COROLLARY 5.8. *Let T be a paracompact Hausdorff space, $X=\mathscr{C}_b(T, U)$, and $\mathscr{F}\subset CB(X)$ be as in Theorem 5.6. Then $\mathscr{C}_b(T, V)$ has r.c.p. for $\mathscr{F}$ in each of the following cases*:

(1) *V is a boundedly compact and convex subset of a Banach space U;*

(2) *$U=l_1$ and V is a w*-closed convex subset of l_1;*

(3) *U is locally uniformly convex, $\mathscr{F}=K(X)$, $V\in CC(U)$, and V satisfies r.c.p. for $K(U)$.*

THEOREM 5.9. *Let T be a paracompact Hausdorff space, U be a Banach space, and $X=\mathscr{C}_b(T, U)$. Let $V\in CC(X)$ and $\mathscr{F}\subset CB(X)$ be such that*

(1) *$f\in\mathscr{C}_b(T, U)$, $f(t)\in\overline{V(t)}$ for each $t\in T$ imply $f\in V$;*

(2) *for each $F\in\mathscr{F}$, the set-valued map $t\to\overline{F(t)}$ of T into $CB(U)$ is u.H.s.c.;*

(3) *the triplet $(U, \mathscr{V}, CB(U))$ satisfies property $(\tilde{R}_4)$, where $\mathscr{V}:=\{\overline{V(t)}: t\in T\}$.*

Then V has r.c.p. for $\mathscr{F}$.

Proof. Let $F\in\mathscr{F}$ be given. For each $t\in T$, define

$$\Phi(t):=\{u\in\overline{V(t)}: r(\overline{F(t)}; u)\leqslant \operatorname{rad}_V(F)\}.$$

Since $\operatorname{rad}_{\overline{V(t)}}(\overline{F(t)})\leqslant\operatorname{rad}_V(F)$, by $(\tilde{R}_4)$, $\Phi(t)\neq\phi$; also it is closed and convex. Thus Φ maps T into $CC(U)$. Using property $(\tilde{R}_4)$ in place of (R_2) exactly as in the proof of Theorem 5.6, we conclude that Φ is l.s.c. Therefore, by Michael's selection theorem [18], Φ has a continuous selection h. Clearly $h\in\mathscr{C}_b(T, U)$ and by (1) $h\in\operatorname{Cent}_V(F)$. Thus V has r.c.p. for $\mathscr{F}$. ∎

COROLLARY 5.10. *Let T be a compact Hausdorff space, U be a Banach space, and let $X=\mathscr{C}(T, U)$. If V is a closed $\mathscr{C}(T, \mathscr{K})$-submodule of X and the triplet $(U, \mathscr{V}, K(U))$ satisfies property $(\tilde{R}_4)$, where $\mathscr{V}=\{\bar{V}(t): t\in T\}$, then V has r.c.p. for $K(X)$.*

Proof. By [19, Approximation Lemma 3.0], (1) in Theorem 5.9 is satisfied. Also by Lemma 1.1, (2) is fulfilled for each $F\in K(X)$ and the conclusion follows from the last theorem. ∎

Last, from Corollary 4.7 (1) and (6), Corollary 3.14, and the preceding Corollary, we obtain

COROLLARY 5.11. *Let T be a compact Hausdorff space, $X=\mathscr{C}(T, U)$, and V be a closed $\mathscr{C}(T, \mathscr{K})$-submodule of X. Then V has r.c.p. for $K(X)$ in each of the following cases*:

(1) *U is a Lindenstrauss space and* $\overline{V(t)}$ *is an M-ideal in U for each* $t \in T$;

(2) *S is an arbitrary set,* $U = l_\infty(S, \mathscr{R})$, *and* $\overline{V(t)}$ *is a linear subspace of U satisfying the condition in Proposition* 2.9 *for each* $t \in T$;

(3) *S is a topological space, E is a uniformly convex Banach space,* $U = l_\infty(S, E)$, *and* $\overline{V(t)}$ *is a* $\mathscr{C}_b(S, \mathscr{K})$*-submodule of* $\mathscr{C}_b(S, E)$ *for each* $t \in T$.

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